Pre-class Warm-up!!!

Let A be the matrix $\begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$

Which of the following vectors v have the property that Av is a scalar multiple of v?

a. $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

b. $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

c. $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ✓

d. None of the above vectors

Is $\begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix}$? No

Is $\begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 1 \end{bmatrix}$? Yes

6.1 Eigenvalues and eigenvectors

New vocabulary:

- eigenvalues and eigenvectors
- the characteristic polynomial of a matrix

We learn:

- how to find the eigenvalues and eigenvectors as the result of a theorem about the characteristic polynomial

What we don't learn:

- why we should be interested in e-values and e-vectors

Definition of an eigenvector and eigenvalue of an n x n matrix A.

We say that a vector v is an eigenvector of A with eigenvalue λ if $Av = \lambda v$ and $v \neq 0$. λ is a number.

Example: Let $A = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$

Try $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad Av = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A with e-value 7.

Try $v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad Av = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

so $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an e-vector with e-value -1.

Definition of the characteristic polynomial of A.

This is the polynomial $\det(A - \lambda I)$ where λ is a variable and I is the $n \times n$ identity matrix.

Example: Let $A = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$. The characteristic polynomial is $\det\left(\begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)$

$= \det\begin{bmatrix} 3-\lambda & 4 \\ 4 & 3-\lambda \end{bmatrix} = (3-\lambda)^2 - 16 = \lambda^2 - 6\lambda - 7$

$= (\lambda - 7)(\lambda + 1)$

Its roots are 7, -1

Theorem. For an n x n matrix A, the eigenvalues of A are precisely the solutions of the characteristic equation

$$\det(A - \lambda I) = 0$$

Proof $\det(A - \lambda I) = 0$

$\iff$ $A - \lambda I$ is not investible

$\iff$ many things e.g. the echelon form as a zero row

$\iff$ there is a free variable

$\iff$ $(A - \lambda I)v = 0$ has non-zero solution v

$\iff$ there is non-zero v with $Av - \lambda I v = 0$ $\iff$ there is a non-zero v with $Av = \lambda v$

$\iff$ there is a non-zero v which is an e-vector with e-value λ.

The numbers λ that appear as e-values are precisely the solutions to the characteristic equation.

To find the eigenvalues of A: Find the roots λ of the characteristic polynomial.

To find the eigenvectors of A:

We solve $Av = \lambda v$ i.e. we solve $(A - I\lambda)v = 0$

i.e. we compute the nullspace of $(A - \lambda I)$

Definition: the eigenspace for the eigenvalue λ is defined to be Null $(A - \lambda I)$

To find the eigenvalues of A: We solve

$\det(A - \lambda I) = 0$

To find the eigenvectors of A: find the eigenvalues. For each e-value λ, solve $Av = \lambda v$, or $(A - \lambda I)v = 0$. Find a basis for $\mathrm{Null}(A - \lambda I)$

Question like 6.1, 1-26.

Find the eigenvalues and eigenvectors of

$A = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$

Solution: Step 1. The characteristic polynomial is $\lambda^2 - 6\lambda - 7 = (\lambda - 7)(\lambda + 1)$

The e-values are $7, -1$

Step 2. We solve $(A - \lambda I)v = 0$.

$\lambda = 7$: $A - \lambda I = \begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix}$

To find the nullspace: Echelon form $\begin{bmatrix} -4 & 4 \\ 0 & 0 \end{bmatrix}$. Basis for nullspace: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = -1$ $A - \lambda I = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$ Echelon form $\begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix}$. Basis for nullspace $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

We obtain two e-values $7, -1$ with e-vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ respectively.

6.1 question 21.

Find the eigenvalues and eigenvectors. Find a basis for each eigenspace of dimension 2 or larger.

$$A = \begin{bmatrix} 4 & -3 & 1 \\ 2 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Solution Char. poly: $\det \begin{bmatrix} 4-\lambda & -3 & 1 \\ 2 & -1-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix}$

$= (2-\lambda) \det \begin{bmatrix} 4-\lambda & -3 \\ 2 & -1-\lambda \end{bmatrix} = (2-\lambda)(\lambda^2 - 3\lambda - 4 + 6)$

$= (2-\lambda)(\lambda^2 - 3\lambda + 2) = -\lambda^3 + 5\lambda^2 - 8\lambda + 4$

Find the roots. Try to find some root

$\lambda = 0$? No $\lambda = 1$ Yes !! $\lambda - 1$ is a factor

$(\lambda^3 - 5\lambda^2 + 8\lambda - 4) = (\lambda - 1)(\lambda^2 - 4\lambda + 4)$

$= (\lambda - 1)(\lambda - 2)^2$

Find the eigenspaces:

$\lambda = 2$ find the nullspace of

$$A - 2I = \begin{bmatrix} 2 & -3 & 1 \\ 2 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Echelon form $\begin{bmatrix} 2 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Two free variables.

Basis for the $\lambda = 2$ e-space

$$\begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3/2 \\ 1 \\ 0 \end{bmatrix}$$

For $\lambda = 1$ we get an e-vector $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Question: For the identity matrix $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

1. How many distinct eigenvalues does I have? 1 occurs with mult. 3, Answr 1.
2. How many independent eigenvectors can we find for I? Answr. 3

a. 0

b. 1

c. 2

d. 3

e. 4 or more

Another question: Same questions 1 and 2 for the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.